Generalized Parametrization Dependence in Quantum Gravity

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We critically examine the gauge, and field-parametrization dependence of renormalization group flows in the vicinity of non-Gaußian fixed points in quantum gravity. While physical observables are independent of such calculational specifications, the construction of quantum gravity field theories typically relies on off-shell quantities such as β functions and generating functionals and thus face potential stability issues with regard to such generalized parametrizations. We analyze a two-parameter class of covariant gauge conditions, the role of momentum-dependent field rescalings and a class of field parametrizations. Using the product of Newton and cosmological constant as an indicator, the principle of minimum sensitivity identifies stationary points in this parametrization space which show a remarkable insensitivity to the parametrization. In the most insensitive cases, the quantized gravity system exhibits a non-Gaußian UV stable fixed point, lending further support to asymptotically free quantum gravity. One of the stationary points facilitates an analytical determination of the quantum gravity phase diagram and features ultraviolet and infrared complete RG trajectories with a classical regime.

I. INTRODUCTION

Physical observables are independent of their computational derivation. Still, many practical computations are based on convenient choices for intermediate auxiliary tools such as coordinate systems, gauges, etc. Appropriate parametrizations of the details of a system simply decrease the computational effort. Beyond pure efficiency aspects, such suitable parametrizations can also be conceptually advantageous or even offer physical insight. This is similar to coordinate choices in classical mechanics where polar coordinates with respect to the ecliptic plane in celestial mechanics support a better understanding in comparison with, say, Cartesian coordinates with a z axis pointing towards Betelgeuse.

Appropriate parameterizations become particularly significant in quantum calculations. While on-shell quantities such as S-matrix elements are invariant observables [1–3], off-shell quantities generically feature parametrization dependencies, including gauge-, fieldparametrization and regularization-scheme dependencies [4–6]. Further ordering schemes such as perturbative expansions may defer such dependencies to higher orders (such as scheme dependence in mass-independent schemes), but these are merely special and not always useful limits. Approximation schemes that can also deal with non-perturbative regimes may even introduce further artificial parametrization dependencies which have to be carefully removed (e.g., discretization artefacts in lattice regularizations).

In an ideal situation, this parametrization dependence of a nonperturbative approximation could be quantified and proven to be smaller than the error of the truncated solution. However, as soon as a result is parametrization dependent, it is likely that some pathological parametrization can be constructed that modifies the result in an arbitrary fashion. This suggests to look for general criteria of good parametrizations that minimize the artificial dependence in approximation schemes which adequately capture the physical mechanisms.

A-priori criteria suggest the construction of parametrizations that support the identification of physically relevant degrees of freedom, such as the use of Coulomb-Weyl gauge in quantum optics, or the use of pole-mass regularization schemes in heavy-quark physics. Further a-priori criteria include symmetry preserving properties (covariant gauges, non-linear field parametrizations) or strict implementations of a parametrization condition such as the Landau-gauge limit $\alpha \to 0$. A major advantage of the latter is that some redundant degrees of freedom decouple fully from the dynamical equations in such a limit.

Good parametrizations may also be identified a posteriori by allowing for a family of parametrizations and identifying stationary points in the parameter space. This realizes the principle of minimum sensitivity [7, 8] (originally advocated for regularization-scheme dependencies), suggesting those points as candidate parameters for minimizing the influence of parametrization dependencies.

In the present work, we investigate a two-parameter family of covariant gauges, a family of field parametrizations and the role of momentum-dependent field rescalings in quantum gravity in this spirit. The family of gauges includes a (non-harmonic) generalization of the harmonic gauge (De-Donder gauge), the latter being particularly useful for the analysis of gravitational waves which presumably are the asymptotic states of quantum gravity. The a-priori criteria suggest to implement this gauge in the Landau-gauge limit to decouple a redundant part of the Hilbert space. In fact, in this limit we find a subtlety in the form of a degeneracy in the subspace of scalar field components which is special to gravity.

We also investigate a one-parameter family of field parametrizations that includes the most widely used linear split [9] as well as the exponential split [10–14] studied more recently in the context of asymptotic safety [15–18] – both of which find support by discriminative a-priori arguments. We also take a brief look at the most general ultra-local four-parameter family of parametrizations to

quadratic order, corroborating the results of the oneparameter family. In addition, we study the influence of momentum-dependent field rescalings which are commonly used in gravity in connection with the York decomposition. In the context of the functional renormalization group (FRG) [19] which provides a tool to study quantum gravity nonperturbatively [20], these parametrization dependencies can mix nontrivially with the regularization of the spectrum of fluctuations. Therefore, the analysis of parametrization dependencies also explores implicitly the stability of the system in the ultraviolet.

Interestingly, we observe a nontrivial interplay between all these parametrization dependencies. Still, several stationary points can be observed in the results for the RG flow where the system develops a remarkable insensitivity to the details of the parametrization choices. In particular, for the stable parametrizations, we observe the existence of a UV stable non-Gaußian fixed point which provides further quantitative evidence for the existence of an asymptotically safe metric quantum gravity [21, 22]. In the stationary regime of the parametrization based on the exponential split, the resulting RG flow exhibits several remarkable properties: (1) a possible dependence on the residual gauge parameter drops out implying an enhanced degree of gauge invariance, (2) the RG flow becomes particularly simple, such that the phase diagram in the plane of Newton and cosmological constant can be computed analytically, (3) no singularities arise in the flow, such that a large class of RG trajectories (including those with a classical regime) can be extended to arbitrarily high and low scales, (4) the UV critical exponents are real and close to their canonical counterparts, and (5) indications are found that the asymptotic safety scenario may not extend straightforwardly to dimensions much higher than d=4.

II. QUANTUM GRAVITY AND PARAMETRIZATIONS

The technical goal of quantum gravity is to construct a functional integral over suitable integration variables which in the long-range limit can be described by a diffeomorphism-invariant effective field theory of metric variables approaching a classical regime for a wide range of macroscopic scales. The fact that the first part of this statement is rather unspecific is reflected by the large number of legitimate quantization proposals [23–25]. Independently of the precise choice of integration variables, a renormalization group approach appears useful in order to facilitate a scale-dependent description of the system and a matching to the long-range classical limit which is given at least to a good approximation by an (effective) action of Einstein-Hilbert type:

$$\Gamma_k = -\int d^d x \sqrt{g} \, \mathcal{Z}_R(R - 2\Lambda). \tag{1}$$

Here, we have already introduced a momentum scale k, expressing the fact that this effective description should a priori hold only for a certain range of classical scales. In this regime, we have $\mathcal{Z}_R=1/(16\pi G_{\rm N})$ with the Newton constant $G_{\rm N}$, and Λ parametrizing the cosmological constant. In a quantum setting, \mathcal{Z}_R plays the role of a (dimensionful) wave-function renormalization, and $G_{\rm N}$ and Λ are expected to be replaced by their running counterparts depending on the scale k.

In the present work, we confine ourselves to a quantum gravity field theory assuming that the metric itself is already a suitable integration variable. A first step towards a diffeomorphism-invariant functional integral then proceeds via the Faddeev-Popov method involving a gauge choice for intermediate steps of the calculation. In this work, we use the background-field gauge with the gauge-fixing quantity,

$$F_{\mu} = \left(\delta^{\beta}_{\mu}\bar{D}^{\alpha} - \frac{1+\beta}{d}\bar{g}^{\alpha\beta}\bar{D}_{\mu}\right)g_{\alpha\beta},\tag{2}$$

which should vanish if the gauge condition is exactly matched. Here, $g_{\alpha\beta}$ is the full (fluctuating) metric, whereas $\bar{g}_{\alpha\beta}$ denotes a fiducial background metric which remains unspecified, but assists to keep track of diffeomorphism invariance within the background-field method. Gauge-fixing is implemented in the functional integral by means of the gauge-fixing action

$$\Gamma_{\rm gf} = \frac{\mathcal{Z}_R}{2\alpha} \int d^d x \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_{\mu} F_{\nu}. \tag{3}$$

More precisely, this gauge choice defines a two-parameter (α,β) family of covariant gauges. For instance, the choice $\beta=1$ corresponds to the harmonic/De-Donder gauge which together with $\alpha=1$ (Feynman gauge) yields a variety of technical simplifications, being used in standard effective field theory calculations [26–28] as well as in functional RG studies [20, 29] of quantum gravity. More conceptually, the Landau-gauge limit $\alpha\to 0$ appears favorable, as it implements the gauge condition in a strict fashion and thus should be a fixed point under RG evolution [30, 31].

In the Euclidean formulation considered here, the parameter α is bound to be non-negative to ensure the positivity of the gauge-fixing part of the action (this restriction may not be necessary for a Lorentzian formulation). The parameter β can be chosen arbitrarily except for the singular value $\beta_{\rm sing} = d-1$. To elucidate this singularity, let us take a closer look at the induced Faddeev-Popov ghost term:

$$\Gamma_{\rm gh} = -\int d^d x \sqrt{\bar{g}} \bar{C}_{\mu} \mathcal{M}^{\mu}_{\ \nu} C^{\nu}, \quad \mathcal{M}^{\mu}_{\ \nu} = \frac{\delta F^{\mu}}{\delta v^{\nu}}, \quad (4)$$

where v^{ν} characterizes the vector field along which we study the Lie derivative generating the coordinate transformations,

$$\frac{\delta g_{\alpha\beta}}{\delta v^{\nu}} = \frac{\delta}{\delta v^{\nu}} \mathcal{L}_{v} g_{\alpha\beta} = 2 \frac{\delta}{\delta v^{\nu}} D_{(\alpha} v_{\beta)}. \tag{5}$$

The corresponding variation of the gauge-fixing condition yields

$$\delta F^{\mu} = 2 \left(\bar{g}^{\mu\alpha} \bar{D}^{\beta} - \frac{(1+\beta)}{d} \bar{g}^{\alpha\beta} \bar{D}^{\mu} \right) D_{(\alpha} \delta v_{\beta)}. \tag{6}$$

Let us decompose the vector δv_{β} into a transversal part $\delta v_{\beta}^{\rm T}$ and a longitudinal part $D_{\beta}\delta\chi$. For the following argument, it suffices to study the limit of the quantum metric approaching the background metric $g_{\mu\nu} \to \bar{g}_{\alpha\beta}$, which diagrammatically corresponds to studying the inverse ghost propagator ignoring higher vertices

$$\delta F^{\mu} = (\delta^{\mu}_{\nu} \bar{D}^{2} + \bar{R}^{\mu}_{\nu}) \delta v^{\mathrm{T}\nu}
+ \frac{1}{2} ((d - 1 - \beta) \bar{D}^{\mu} \bar{D}_{\nu} + 4 \bar{R}^{\mu}_{\nu}) \bar{D}^{\nu} \delta \chi + \mathcal{O}(g - \bar{g}).$$
(7)

In this form it is obvious that the longitudinal direction $\bar{D}^{\nu}\delta\chi$ is not affected by the gauge fixing for $\beta=d-1$ to zeroth order in the curvature. In other words, the gauge fixing is not complete for this singular case $\beta_{\rm sing}=d-1$. This singularity is correspondingly reflected by the ghost propagator. The Faddeev-Popov operator in Eq. (4) reads

$$\mathcal{M}^{\mu}_{\ \nu} = 2\bar{g}^{\mu\beta}\bar{D}^{\alpha}D_{(\alpha}g_{\beta)\nu} - 2\frac{1+\beta}{d}\bar{g}^{\alpha\beta}\bar{D}^{\mu}D_{\alpha}g_{\beta\nu}. \quad (8)$$

Decomposing the ghost fields \bar{C}_{μ} , C^{ν} also into transversal $\bar{C}_{\mu}^{\mathrm{T}}$, $C^{\mathrm{T}\nu}$ and longitudinal parts $\bar{D}^{\mu}\bar{\eta}$, $\bar{D}^{\nu}\eta$ we find for the ghost Lagrangian,

$$\bar{C}_{\mu}\mathcal{M}^{\mu}_{\ \nu}C^{\nu} = \bar{C}_{\mu}^{\mathrm{T}} \left(\delta_{\nu}^{\mu} \bar{D}^{2} + \bar{R}_{\nu}^{\mu} \right) C^{\mathrm{T}\nu}
- \bar{\eta} \left(\frac{d - 1 - \beta}{2} \bar{D}^{4} + \bar{R}^{\mu\nu} \bar{D}_{\mu} \bar{D}_{\nu} \right) \eta + \mathcal{O}(g - \bar{g}).$$
(9)

where we have performed partial integrations in order to arrive at a convenient form and dropped covariant derivatives of the curvature. This form of the inverse propagator of the ghosts makes it obvious that a divergence of the form $\frac{1}{d-1-\beta}$ arises in the longitudinal parts. This divergence at $\beta_{\rm sing} = d-1$ related to an incomplete gauge fixing will be visible in all our results below.

Let us now turn to the metric modes. As a technical tool, we parametrize the fully dynamical metric $g_{\mu\nu}$ in terms of a fiducial background metric $\bar{g}_{\mu\nu}$ and fluctuations $h_{\mu\nu}$ about the background. Background independence is obtained by keeping $\bar{g}_{\mu\nu}$ arbitrary and requiring that physical quantities such as scattering amplitudes are independent of $\bar{g}_{\mu\nu}$. Still, these requirements do not completely fix the parametrization of the dynamical field $g=g[\bar{g};h]$. Several parametrizations have been used in concrete calculations. The most commonly used parametrization is the linear split [9]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,. \tag{10}$$

By contrast, the exponential split [10–14]

$$g_{\mu\nu} = \bar{g}_{\mu\rho} \left(e^h \right)^{\rho}_{\ \nu},\tag{11}$$

is a parametrization that has been discussed more recently to a greater extent [15–18]. In both cases, h is considered to be a symmetric matrix field (with indices raised and lowered by the background metric). If a path integral of quantum gravity is now defined by some suitable measure $\mathcal{D}h$, it is natural to expect that the space of dynamical metrics g is sampled differently by the two parametrizations, implying different predictions at least for off-shell quantities – unless the variable change from (10) to (11) is taken care of by suitable (ultralocal) Jacobians. While a parametrization (and gauge-condition) independent construction of the path integral has been formulated in a geometric setting [9, 32–35], its usability is hampered by the problem of constructing the full decomposition of h in terms of fluctuations between physically inequivalent configurations and fluctuations along the gauge orbit. Geometric functional RG flows have been conceptually developed in [36], with first results for asymptotic safety obtained in [37], and recently to a leading-order linear-geometric approximation in [38]. The relation between the geometric approach and the exponential parametrization was discussed in [16].

In the present work, we take a more pragmatic view-point, and consider the different parameterizations of Eqs. (10) and (11) as two different approximations of an ideal parametrization. Since the functional RG actually requires the explicit form of $g[\bar{g};h]$ only to second order in h (in the single-metric approximation, see below), we mainly consider a one-parameter class of parametrizations of the type

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \frac{\tau}{2} h_{\mu\rho} h_{\nu}^{\rho} + \mathcal{O}(h^3).$$
 (12)

For $\tau=0$, we obtain the linear split, whereas $\tau=1$ is exactly related to the exponential split within our truncation. Incidentally, it is straightforward to write down the most general, ultra-local parametrization to second order that does not introduce a scale,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \frac{1}{2} \left(\tau h_{\mu\rho} h^{\rho}_{\nu} + \tau_2 h h_{\mu\nu} + \tau_3 \bar{g}_{\mu\nu} h_{\rho\sigma} h^{\rho\sigma} + \tau_4 \bar{g}_{\mu\nu} h^2 \right) + \mathcal{O}(h^3).$$
 (13)

Here, $h=h^{\mu}_{\mu}$ is the trace of the fluctuation. As mentioned above, third and higher-order terms will not contribute to our present study anyway. Instead of exploring the full parameter dependence, we will highlight some interesting results in this more general framework below.

The key ingredient for a quantum computation is the propagator of the dynamical field. In our setting, its inverse is given by the second functional derivative (Hessian) of the action (1) including the gauge fixing (3) with respect to the fluctuating field h,

$$\begin{split} &\frac{1}{\mathcal{Z}_{R}} \left. \Gamma_{hh}^{(2)\kappa\nu}{}_{\alpha\beta} \right|_{h=0,C=0} \\ &= \frac{1}{16\alpha} \left(8\alpha \delta_{\alpha\beta}^{\kappa\nu} - [8\alpha - (1+\beta)^{2}] \bar{g}^{\kappa\nu} \bar{g}_{\alpha\beta} \right) (-\bar{D}^{2}) \\ &- \frac{1-\alpha}{\alpha} \delta_{(\alpha}^{(\kappa} \bar{D}^{\nu)} \bar{D}_{\beta)} \\ &+ \frac{1+\beta-2\alpha}{4\alpha} \left(\bar{g}^{\kappa\nu} \bar{D}_{(\alpha} \bar{D}_{\beta)} + \bar{g}_{\alpha\beta} \bar{D}^{(\kappa} \bar{D}^{\nu)} \right) \\ &+ \frac{1}{4} \left(2(1-\tau) \delta_{\alpha\beta}^{\kappa\nu} - \bar{g}^{\kappa\nu} \bar{g}_{\alpha\beta} \right) (\bar{R} - 2\lambda_{k}) \\ &- (1-\tau) \bar{R}_{(\alpha}^{(\kappa} \delta_{\beta)}^{\nu)} + \frac{1}{2} (\bar{R}^{\kappa\nu} \bar{g}_{\alpha\beta} + \bar{R}_{\alpha\beta} \bar{g}^{\kappa\nu}) - \bar{R}^{\kappa}_{(\alpha}{}_{\beta)}^{\nu}, \end{split}$$

Here and in the following, we specialize to d=4, except if stated otherwise. A standard choice for the gauge parameters is harmonic DeDonder gauge with $\alpha=1=\beta$ for which the second and third lines simplify considerably. Simplifications also arise for the exponential split $\tau=1$; in particular, a dependence on the cosmological constant λ_k remains only in the trace mode $\sim \bar{g}^{\kappa\nu}\bar{g}_{\alpha\beta}$.

A standard tool for dealing with the tensor structure of the propagator is the York decomposition of the fluctuations $h_{\mu\nu}$ into transverse traceless tensor modes, a transverse vector mode and two scalar modes,

$$h_{\mu\nu} = h_{\mu\nu}^{\mathrm{T}} + 2\bar{D}_{(\mu}\xi_{\nu)}^{\mathrm{T}} + \left(2\bar{D}_{(\mu}\bar{D}_{\nu)} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{D}^{2}\right)\sigma + \frac{1}{4}\bar{g}_{\mu\nu}h,$$
(15)

$$\bar{D}^{\mu}h_{\mu\nu}^{\mathrm{T}} = 0, \quad \bar{g}^{\mu\nu}h_{\mu\nu}^{\mathrm{T}} = 0, \quad \bar{D}^{\mu}\xi_{\mu}^{\mathrm{T}} = 0.$$
 (16)

It is convenient to split $\Gamma^{(2)}$ into a pure kinetic part \mathcal{P} which has a nontrivial flat-space limit, and a curvature-dependent remainder $\mathcal{F} = \mathcal{O}(\bar{R})$. This facilitates an expansion of the propagator $(\Gamma^{(2)})^{-1} = (\mathcal{P} + \mathcal{F})^{-1} = \sum_{n=0}^{\infty} (-\mathcal{P}^{-1}\mathcal{F})^n \mathcal{P}^{-1}$.

Let us first concentrate on the kinetic part \mathcal{P} :

$$\mathcal{P}_{h^{\mathrm{T}} \alpha\beta}^{\mu\nu} = \frac{\mathcal{Z}_{R}}{2} \delta_{\alpha\beta}^{\mu\nu} (\Delta - 2(1-\tau)\lambda_{k}), \tag{17}$$

$$\mathcal{P}_{\xi^{\mathrm{T}}\alpha}^{\mu} = \frac{\mathcal{Z}_{R}}{\alpha} \delta_{\alpha}^{\mu} \Delta \left(\Delta - 2\alpha (1 - \tau) \lambda_{k} \right), \tag{18}$$

$$\mathcal{P}_{(\sigma h)} = \mathcal{Z}_R \begin{pmatrix} 3 \frac{(3-\alpha)\Delta - 4\alpha(1-\tau)\lambda_k}{4\alpha} \Delta^2 & \frac{3}{8\alpha} (\beta - \alpha)\Delta^2 \\ \frac{3}{8\alpha} (\beta - \alpha)\Delta^2 & \frac{(\beta^2 - 3\alpha)\Delta + 4\alpha(1+\tau)\lambda_k}{16\alpha} \end{pmatrix}, \tag{19}$$

where $\Delta = -\bar{D}^2$. In this form it is straightforward to calculate the propagator $(\mathcal{P})^{-1}$. In particular, the transverse traceless mode h^{T} does not exhibit any dependence on the gauge parameters. As discussed in the introduction, a-priori criteria suggest the Landau-gauge limit $\alpha \to 0$ as a preferred choice for the gauge fixing, as it strictly implements the gauge-fixing condition. It thus should also be a fixed point of the RG flow [30, 31].

Whereas the choice of α and β , in principle, are independent, there can arise a subtle interplay with certain regularization strategies as will be highlighted in the following.

By taking the limit $\alpha \to 0$ while keeping β finite, we make the gauge fixing explicit, especially we find for the gauge-dependent modes

$$\mathcal{P}_{\xi^{\mathrm{T}}\alpha}^{-1\mu} \to \alpha \frac{1}{\mathcal{Z}_{\mathrm{P}}\Lambda^2} \delta_{\alpha}^{\mu},\tag{20}$$

$$\mathcal{P}_{(\sigma h)}^{-1} \to \frac{-\frac{1}{3Z_R} \Delta^{-2}}{\frac{(3-\beta)^2}{4} \Delta - (3-\beta^2 + (3+\beta^2)\tau)\lambda_k} \begin{pmatrix} \beta^2 & -6\beta\Delta \\ -6\beta\Delta & 36\Delta^2 \end{pmatrix}.$$
(21)

The transverse mode $\xi_{\mu}^{\rm T}$ decouples linearly with $\alpha \to 0$ and hence is pure gauge in the present setting. Whereas finite parts seem to remain in the (σh) subspace, we observe that the matrix $\mathcal{P}_{(\sigma h)}^{-1}$ in (21) becomes degenerate in this limit (e.g., the determinant of the matrix in Eq. (21) is zero). Effectively, only one scalar mode remains in the propagator. The nature of this scalar mode is a function of the second gauge parameter: taking the limit $\beta \to \infty$, the remaining scalar mode can be identified with σ , while the limit $\beta \to 0$ leaves us with a pure h mode.

Whereas the transverse modes in Eq. (20) decouple smoothly in the limit $\alpha \to 0$, the decoupling of the scalar mode in Eq. (21) is somewhat hidden in the degeneracy of the scalar sector with the corresponding eigenmode depending on β . This can lead to a subtle interplay with regularization techniques for loop diagrams as can be seen on rather general grounds by the following argument. Structurally, the propagator in the (σh) sector has the following form in the limit $\alpha \to 0$ and for small but finite β , cf. Eq. (21)

$$\left(\mathcal{P}_{(\sigma h)}\right)^{-1} \to \left(\begin{matrix} \mathcal{O}(\beta^2) & \mathcal{O}(\beta) \\ \mathcal{O}(\beta) & \mathcal{O}(1) \end{matrix}\right).$$
 (22)

Regularizations of traces over loops built from this propagator are typically adjusted to the spectrum of the involved operators. Let us formally write this as

$$\operatorname{Tr}\left[\mathcal{L}_{\mathcal{R}}\mathcal{P}^{-1}(\dots)\right]$$
 (23)

where $\mathcal{L}_{\mathcal{R}}$ denotes a regularizing operator and the ellipsis stands for further vertices and propagators. Now, it is often useful to regularize all fluctuation operators at the same scale, e.g., the spectrum of all Δ 's should be cut off at one and the same scale k^2 . Therefore, the regularizing operator $\mathcal{L}_{\mathcal{R}}$ inherits its tensor structure from the Hessian $\Gamma^{(2)}$ of Eq. (14). In the (σh) sector, the regularizing operator can hence acquire the same dependence on the gauge-parameters as in Eq. (19),

$$\mathcal{L}_{\mathcal{R},(\sigma h)} \to \frac{1}{\alpha} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\beta) \\ \mathcal{O}(\beta) & \mathcal{O}(\beta^2) \end{pmatrix},$$
 (24)

for $\alpha \to 0$ and small β . The complete scalar contribution to traces of the type (23) would then be of the parametric

form,

$$\operatorname{Tr}\left[\mathcal{L}_{\mathcal{R}}\mathcal{P}^{-1}(\dots)\right]_{(\sigma h)} \to \frac{1}{\alpha}\mathcal{O}(\beta^2).$$
 (25)

For finite β , such regularized traces can thus be afflicted with divergencies in the Landau-gauge limit $\alpha \to 0$. If this happens, we still have the option to choose suitable values of β . In fact, Eq. (25) suggest that still a whole one-parameter family of gauges exists in the Landau-gauge limit, if we set $\beta = \gamma \cdot \sqrt{\alpha}$, with arbitrary real but finite gauge parameter γ distinguishing different gauges.

We emphasize that this is a rather qualitative analysis. Since the limit of products is not necessarily equal to the product of limits, the trace over the matrix structure of the above operator products can still eliminate this $1/\alpha$ divergence, such that any finite value of β remains admissible.

In the following we observe that the appearance of the $1/\alpha$ divergence depends on the explicit choice of the regularization procedure, as expected. Still, as this discussion shows, even if this divergence occurs, it can perfectly well be dealt with by choosing $\beta = \gamma \sqrt{\alpha}$ and still retaining a whole one-parameter family of gauges in the Landau gauge limit.

III. GRAVITATIONAL RG FLOW

For our study of generalized parametrization dependencies of gravitational RG flows, we use the functional RG in terms of a flow equation for the effective average action (Wetterich equation) [19] amended by the background-field method [39–41] and formulated for gravity [20]

$$\partial_t \Gamma_k[g, \bar{g}] = \frac{1}{2} \mathrm{STr} \left[\partial_t \mathcal{R}_k \left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} \right], \qquad \partial_t = k \frac{d}{dk}.$$
(26)

Equation (26) describes the flow of an action functional Γ_k as a function of an RG scale k that serves as a regularization scale for the infrared fluctuations. Here, $\Gamma_k^{(2)}$ denotes the Hessian of the action with respect to the fluctuation field g, at fixed background \bar{g} . The details of the regularization are encoded in the choice of the regulator \mathcal{R}_k . Suitable choices of \mathcal{R}_k guarantee that Γ_k becomes identical to the full quantum effective action in the limit $k \to 0$, and approaches the bare action for large scales $k \to \Lambda_{\mathrm{UV}} \to \infty$ (where Λ_{UV} denotes a UV cutoff). For reviews in the present context, see [29, 42–47].

Whereas exact solutions of the flow equation so far have only been found for simple models, approximate nonperturbative flows can be constructed with the help of systematic expansion schemes. In the case of gravity, a useful scheme is given by expanding Γ_k in powers of curvature invariants. The technical difficulties then lie in the construction of the inverse of the regularized Hessian $(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}$, corresponding to the regularized propagator, and performing the corresponding traces (the super-

trace STr includes a minus sign for Grassmann degrees of freedom, i.e., Faddeev-Popov ghosts).

A conceptual difficulty lies in the fact that $\Gamma_k[g,\bar{g}]$ should be computed on a subspace of action functionals that satisfy the constraints imposed by diffeomorphism invariance and background independence. In general, this requires to work with q and \bar{q} independently during large parts of the computation [48–51]. Such bi-metric approaches can, for instance, be organized in the form of a vertex expansion on a flat space as put forward recently in [52-54], or via a level expansion as developed in [55], see [56–58] for further bi-metric results. For the present study of parametrization dependencies, we confine ourselves to a single-metric approximation, defined by identifying g with \bar{g} on both sides of the flow equation, after the Hessian has been analytically determined. In the following, we therefore do no longer have to distinguish between the background field and the fluctuation field as far as the presentation is concerned, and hence drop the bar notation for simplicity.

Spanning the action in terms of the Einstein-Hilbert truncation (1) and neglecting the flow of the gauge-fixing and ghost sector [59–61], we use the universal RG machine [62–64] as our computational strategy. The key idea is to subdivide the Hessian $\Gamma_k^{(2)}$ into a kinetic part and curvature parts with a subsequent expansion in the curvature. This is complicated by terms containing uncontracted covariant derivatives in $\Gamma_k^{(2)}$ which could invalidate the counting scheme. Within the present truncation, this problem is solved with the aid of the York decomposition (15). This helps both to set up the curvature expansion as well as to invert the kinetic terms in the corresponding subspaces of TT, T and scalar modes. From a technical point of view, we use the package xAct [65–70] to handle the extensive tensor calculus.

Schematically, the flow equation for the Einstein-Hilbert truncation can then be written as

$$\partial_t \Gamma_k = \int d^4 x \sqrt{-g} \left(\mathcal{S}^{\text{TT}} + \mathcal{S}^{\text{T}} + \mathcal{S}^{\sigma h} + \mathcal{S}^{\text{gh}} + \mathcal{S}^{\text{Jac}} \right),$$
(27)

where the first three terms denote the contributions from the graviton fluctuations as parametrized by the York decomposition (15). The fourth term $\mathcal{S}^{\mathrm{gh}}$ arises from the Faddeev-Popov ghost fluctuations, cf. Eq. (10). The last term $\mathcal{S}^{\mathrm{Jac}}$ comes from the use of transverse decompositions of the metric (15) and the ghost fields (10). The corresponding functional integral measure over the new degrees of freedom involves Jacobians which – upon analogous regularization – contribute to the flow of the effective average action.

At this point, we actually have a choice that serves as another source of parametrization dependencies studied in this work: one option is to formulate the regularized path integral in terms of the decomposed fields as introduced above. In that case, the Jacobians are nontrivial and their contribution \mathcal{S}^{Jac} is listed in Eq. (A21). Alternatively, we can reintroduce canonically normalized fields

by means of a nonlocal field redefinition [71, 72],

$$\sqrt{\Delta - \text{Ric}} \ \xi^{\mu} \to \xi^{\mu},$$
 (28)

$$\sqrt{\Delta - \text{Ric}} \, \xi^{\mu} \to \xi^{\mu}, \tag{28}$$

$$\sqrt{\Delta^2 + \frac{4}{3} D_{\mu} R^{\mu\nu} D_{\nu}} \, \sigma \to \sigma, \tag{29}$$

$$\sqrt{\Delta} \eta \to \eta,$$
 (30)

and analogously for the longitudinal anti-ghost field $\bar{\eta}$. (Here, we have used $(\operatorname{Ric} \xi)^{\mu} = R^{\mu\nu} \xi_{\nu}$.) This field redefinition goes along with another set of Jacobians contributing to the measure of the rescaled fields. As shown in [72], the Jacobians for the original York decomposition and the Jacobians from the field redefinition (30) cancel at least on maximally symmetric backgrounds. The latter choice of backgrounds is sufficient for identifying the flows in the Einstein-Hilbert truncation. Therefore, if we set up the flow in terms of the redefined fields (30), the last term in Eq. (27) vanishes, $S_{fr}^{Jac} = 0$.

For an exact solution of the flow, it would not matter whether or not a field redefinition of the type (30) is performed. Corresponding changes in the full propagators would be compensated for by the (dis-)appearance of the Jacobians. For the present case of a truncated nonperturbative flow, a dependence on the precise choice will, however, remain, which is another example for a parametrization dependence. This dependence also arises from the details of the regularization. The universal RG machine suggests to construct a regulator \mathcal{R}_k such that the Laplacians Δ appearing in the kinetic parts are replaced by

$$\Delta \to \Delta + R_k(\Delta),$$
 (31)

where $R_k(x)$ is a (scalar) regulator function that provides a finite mass-like regularization for the long-range modes, e.g., $R_k(x) \to k^2$, for $x \ll k^2$, but leaves the UV modes unaffected, $R_k(x) \to 0$ for $x \gg k^2$. Since the field redefinition (30) is nonlocal, it also affects the kinetic terms and thus takes influence on the precise manner of how modes are regularized via Eq. (31). In other words, the dependence of our final results on using or not using the field redefinition (30) is an indirect probe of the regularization-scheme dependence and thus of the generalized parametrization dependence we are most interested in here.

In this work, we focus on the RG flow of the effective average action parametrized by the operators of the Einstein-Hilbert truncation (1). For this, we introduce the dimensionless versions of the gravitational coupling and the cosmological constant,

$$g := \frac{k^2}{16\pi \mathcal{Z}_R} \equiv k^2 G, \quad \lambda = \frac{\Lambda}{k^2}, \tag{32}$$

and determine the corresponding RG β functions for gand λ , by computing the S terms on the right hand side of the flow (27) to order R in the curvature. Many higherorder computations have been performed by now [73–84], essentially confirming and establishing the simple picture visible in the Einstein-Hilbert truncation.

We are particularly interested in the existence of fixed points g_* and λ_* of the β functions, defined by

$$\partial_t g = \dot{g} \equiv \beta_g(g_*, \lambda_*) = 0, \quad \partial_t \lambda = \dot{\lambda} \equiv \beta_\lambda(g_*, \lambda_*) = 0.$$
(33)

In addition to the Gaussian fixed point $g_* = 0 = \lambda_*$, we search for a non-Gausian interacting fixed point, the existence of which is a prerequisite for the asymptoticsafety scenario. Physically viable fixed points should have a positive value for the Newton coupling and should be connectable by an RG trajectory with the classical regime, where the dimensionful couplings are approximately constant, i.e., the dimensionless versions should scale as $g \sim k^2$, $\lambda \sim 1/k^2$. The asymptotic-safety scenario also requires that a possible non-Gausian fixed point has finitely many UV attractive directions. This is quantified by the number of positive critical exponents θ_i which are defined as (-1) times the eigenvalues of the stability matrix $\partial \beta_{(q,\lambda)}/\partial(g,\lambda)$.

Whereas the fixed-point values g_* and λ_* are RG scheme-dependent, the critical exponents θ_i are universal and thus should be parametrization independent in an exact calculation. Also, the product $g_*\lambda_*$ has been argued to be physically observable in principle and thus should be universal [72]. Testing the parametrization dependence of the critical exponents θ_i and $g_*\lambda_*$ therefore provides us with a quantitative criterion for the reliability of approximative results.

IV. GENERALIZED PARAMETRIZATION **DEPENDENCE**

With these prerequisites, we now explore the parametrization dependencies of the following scenarios: we consider the linear (10) and the exponential (11) split, both with and without field redefinition (30), and study the corresponding dependencies on the gauge parameters, focusing on a strict implementation of the gauge-fixing condition $\alpha \to 0$ (Landau gauge). As suggested by the principle of minimum sensitivity, we look for stationary points as a function of the remaining parameter(s) where universal results become most insensitive to these generalized parametrizations. For the following quantitative studies, we exclusively use the piecewise linear regulator [85, 86], $R_k(x) = (k^2 - x^2)\theta(k^2 - x^2)$, for reasons of simplicity. Studies of regulator-scheme dependencies which can also quantify parametrization dependencies have first been performed, e.g., in [72, 78].

Linear split without field redefinition

Let us start with the case of the linear split (10) without field redefinition (30). Here, the degeneracy in the sector of scalar modes interferes with the regularization scheme, as illustrated in Eq. (25). Hence, in the Landaugauge limit $\alpha \to 0$, we choose $\beta = \gamma \sqrt{\alpha}$, which removes

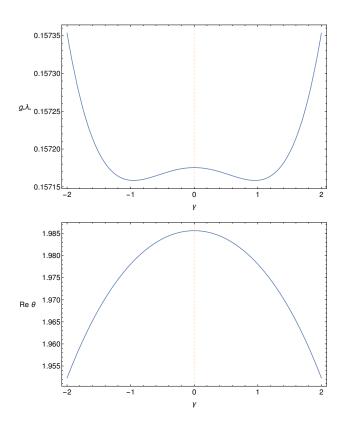


Figure 1. Linear split without field redefinition: residual dependence of our estimates for the universal quantities on gauge parameter γ in the limit $\alpha \to 0$. We find a common stationary point at $\gamma = 0$ and a remarkably small variation of the results on the level of 0.1% for $g_*\lambda_*$ and 1.6% for $\mathrm{Re}\theta$ in the range $\gamma \in [-2, 2]$.

any artificial divergence, but keeps γ as a real parameter that allows for a quantification of remaining parametrization/gauge dependence. We indeed find a non-Gaußian fixed point g_*, λ_* for wide range of values of γ . The critical exponents form a complex conjugate pair. The estimates for the universal quantities $g_*\lambda_*$ and the real part of the θ 's (being the measure for the RG relevance of perturbations about the fixed point) are depicted in Fig. 1.

We observe a common point of minimum sensitivity at $\gamma=0$. In a rather wide range of gauge parameter values $\gamma\in[-2,2]$, our estimates for $g_*\lambda_*$ and $\mathrm{Re}\theta$ vary only very mildy on the level of 0.1% and 1.6%. Given the limitations of the present simple approximation, this is a surprising degree of gauge independence lending further support to the asymptotic-safety scenario. The extremizing values at $\gamma=0$ are near the results of [62, 76, 77] where the same gauge choice $(\alpha=\beta=0)$ was used. The main difference can be traced back to the fact that our inclusion of the (dimensionful) wave-function renormalization in the gauge fixing term (3) renders the gauge parameter α dimensionless as is conventional. If we ignored the resulting dimensional scaling, our extremizing

parametrization	g_*	λ_*	$g_*\lambda_*$	θ
$ \overline{\text{nfr } \tau = \alpha = \gamma = 0} $	0.879	0.179	0.157	$1.986 \pm i \ 3.064$
$ nfr \ \tau = 0, \ \alpha = \beta = 1 $	0.718	0.165	0.119	$1.802 \pm i \ 2.352$
fr $\tau = \alpha = 0, \beta = 1$	0.893	0.164	0.147	$2.034 \pm i \ 2.691$
fr $\tau = 0$, $\alpha = \beta = 1$	0.701	0.172	0.120	$1.689 \pm i \ 2.486$
$fr \ \tau = \alpha = 0, \ \beta = \infty$	0.983	0.151	0.148	$2.245 \pm i \ 2.794$
fr $\tau = 1, \beta = \infty$	3.120	0.331	1.033	4, 2.148
fr $\tau = 1.22$, $\alpha = 0$, $\beta = \infty$	3.873	0.389	1.508	3.957, 1.898

Table I. Non-Gaußian fixed-point properties for several parametrizations, characterized by the gauge parameters α, β or γ , as well as by the choice of the parametrization split parameter τ with $\tau=0$ corresponding to the linear split (10) and $\tau=1$, being the exponential split (11). Whether or not a field redefinition (30) is performed is labeled by "fr" or "nfr", respectively.

result would be exactly that of [62] and in close agreement with [76,77] with slight differences arising from the regularization scheme. We summarize a selection of our quantitative results in Table I.

B. Exponential split without field redefinition

As a somewhat contrary example, let us now study the case of the exponential split (11) also without field redefinition (30). Again, we find a non-Gaußian fixed point. The corresponding estimates for the universal quantities at this fixed point in the Landau gauge limit $\alpha=0$ are displayed in Fig. 2. At first glance, the results seem similar to the previous ones with a stationary point at $\gamma=0$. However, the product $g_*\lambda_*$ shows a larger variation on the order of 5% and the critical exponent even varies by a factor of more than 40 in the range $\gamma \in [-2,2]$. We interpret the strong dependence on the gauge parameter γ as a clear signature that these estimates based on the exponential split without field redefinition should not be trusted.

In fact, the real part of the critical exponents, $Re\theta$. have even changed sign compared to the previous case implying that the non-Gausian fixed point has turned UV repulsive. Similar observations have been made in [15] for the harmonic Feynman-type gauge $\alpha = 1 = \beta$ and an additional strong dependence on the regulator profile function $R_k(x)$ has been found. We have verified that our results agree with those of [15] for the corresponding gauge choice. In summary, this parametrization serves as an example that non-perturbative estimates can depend strongly on the details of the parametrization (even for seemingly reasonable parametrizations) and the results can be misleading. The good news is that a study of the parametrization dependence can – and in this case does - reveal the insufficiency of the parametrization through its strong dependence on a gauge parameter.

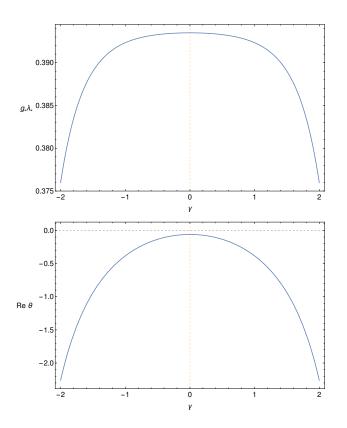


Figure 2. Exponential split without field redefinition: residual dependence of our estimates for the universal quantities on gauge parameter γ in the limit $\alpha \to 0$. A common stationary point is again present at $\gamma = 0$, but the estimates for the universal quantities exhibit a substantial variation in the range $\gamma \in [-2,2]$: $g_*\lambda_*$ varies by $\sim 5\%$ and $\text{Re}\theta$ even by more than a factor of 40. The latter is a clear signal for the insufficiency of the parametrization.

C. Linear split with field redefinition

For the remainder, we consider parametrizations of the fluctuation field which include field redefinitions (30). The canonical normalization achieved by these field redefinitions has not merely aesthetical reasons. An important aspect is that the nonlocal field redefinition helps to regularize the modes in a more symmetric fashion: the kinetic parts of the propagators then become linear in the Laplacian which are all equivalently treated by the regulator (31). A practical consequence is that the interplay of the degeneracy in the scalar sector no longer interferes with the regularization, i.e., the gauge parameter β can now be chosen independently of α . Concentrating again on the Landau-gauge limit $\alpha \to 0$, we observe for generic split parameter τ that $\beta = 0$ no longer is an extremal point.

Our estimates for the universal quantities for the case of the linear split (10) with field redefinition (30) and $\alpha \to 0$ are plotted in Fig. 3. In order to stay away from the singularity at $\beta = 3$, cf. Eq. (10), we consider values

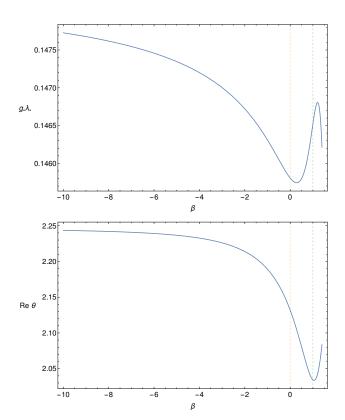


Figure 3. Linear split with field redefinition: residual dependence of our estimates for the universal quantities on the gauge parameter β in the limit $\alpha \to 0$. A common stationary point is approached for $|\beta| \to \infty$. Near the harmonic gauge $\beta = 1$ (green dashed vertical line), both quantities have an extremum. For the whole range of β values, the estimates for the universal quantities exhibit rather small variations of 1% for $g_*\lambda_*$ and 10% for the more sensitive critical exponent Re θ .

for $\beta < 3$ down to $\beta \to -\infty$. As is obvious, e.g., from Eq. (21), the dependence of the propagator of the scalar modes and thus on β is such that the limits of large positive or negative $\beta \to \pm \infty$ yield identical results. Also the longitudinal ghost mode decouples in the limit $\beta \to \pm \infty$ such that the whole flow in the large $|\beta|$ -limit is independent of the sign of β .

A non-Gaußian fixed point exists, and a common extremum of $g_*\lambda_*$ and $\operatorname{Re}\theta$ occurs for $\beta\to-\infty$. Near $\beta=1$ marking the harmonic gauge condition, both quantities are also close to an extremum (which does not occur at exactly the same β value for both quantities). All fixed-point quantities for this case are listed in Tab. I ("fr $\tau=\alpha=0,\ \beta=1$ "). These values agree with the results of [37]. They are remarkably close, e.g., to those for the linear split without field redefinition. The situation is similar for the other extremum $|\beta|\to\infty$ ("fr $\tau=\alpha=0,\ \beta=\infty$ " in Tab. I). For the whole infinite β range studied for this parametrization, $g_*\lambda_*$ varies on the level of 1%. The more sensitive critical exponent $\operatorname{Re}\theta$ varies by 10% which is still surprisingly small given the simplicity of the approximation. Let us emphasize again that

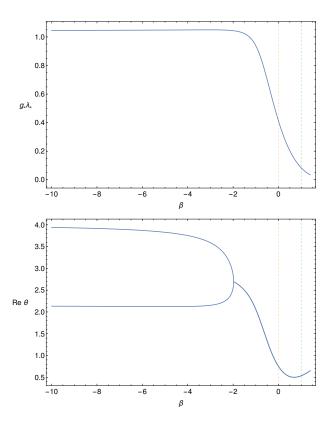


Figure 4. Exponential split with field redefinition: residual dependence of our estimates for the universal quantities on the gauge parameter β in the limit $\alpha \to 0$. A common stationary point is approached for $|\beta| \to \infty$, whereas no common minimum-sensitivity point is found near the harmonic gauge $\beta = 1$ or $\beta = 0$ (dashed vertical lines). Below $\beta \lesssim -2$, the critical exponents become real with the non-Gaußian fixed point remaining UV attractive. For $|\beta| \to \infty$, the results become independent of the gauge parameter α .

varying β from infinity to zero corresponds to a complete exchange of the scalar modes from σ (longitudinal vector component) to h (conformal mode) and hence to a rather different parametrization of the fluctuating degrees of freedom.

D. Exponential split with field redefinition

Finally, we consider the exponential split (10), $\tau=1$, with field redefinition (30). Having performed the latter has a strong influence on the stability of the estimates of the universal quantities at the non-Gaußian fixed point, as is visible in Fig. 4. Contrary to the linear split, we do not find a common extremum near small values of β : neither $\beta=0$ nor the harmonic gauge $\beta=1$ seem special, but, e.g., the product $g_*\lambda_*$ undergoes a rapid variation in this regime.

Rather, a common extremal point is found in the limit $\beta \to \infty$. In fact, $g_*\lambda_*$ becomes insensitive to the precise value of β for $\beta \lesssim -2$ (with a local maximum near $\beta \simeq$

-3, and an asymptotic value of $g_*\lambda_* \simeq 1.033$ for $\beta \to \infty$. This estimate for $g_*\lambda_*$ is significantly larger than for the other parametrizations. The deviation may thus be interpreted as the possible level of accuracy that can be achieved in this simple Einstein-Hilbert truncation.

As an interesting feature, the critical exponents become real for $\beta \lesssim -2$, and approach the asymptotic values $\theta = \{4, 2.148\}$ for $\beta \to \infty$. The leading exponent $\theta = 4$ reflects the power-counting dimension of the cosmological term. This is a straightforward consequence of the fact that the λ dependence in this parametrization $\tau = 1, \beta \to \infty$ disappears from the propagators of the contributing modes. The leading nontrivial exponent $\theta = 2.148$ hence is associated with the scaling of the Newton constant near the fixed point, which is remarkably close to minus the power-counting dimension of the Newton coupling. The latter is a standard result for non-Gaußian fixed points which are described by a quadratic fixed-point equation [87, 88]. The small difference to the value $\theta = 2$ arises from the RG-improvement introduced by the anomalous dimension in the threshold functions (" η -terms" as discussed in the Appendix). Neglecting these terms, the estimate of the leading critical exponents in dimension d is d and d-2, as first discussed in [17]. Also our other quantitative results for the fixed-point properties are in agreement with those of [17] within the same approximation.

The significance of the results within this parametrization is further underlined by the observation that the results in the limit $\beta \to \infty$ become completely independent of the gauge parameter α . In other words, the choice of the transverse traceless mode and the σ mode $(\beta \to \infty)$ as a parametrization of the physical fluctuations removes any further gauge dependence.

The present parametrization has also some relation to [89, 90], where in addition to the exponential split the parametrization was further refined to remove the gauge-parameter dependence completely on the semi-classical level. More specifically, the parametrization of the fluctuations was chosen so that only fluctuations contribute that also have an on-shell meaning. In essence, this removes any contribution from the scalar modes to the UV running. At the semi-classical level [89], the nontrivial critical exponent is 2 as in [17] and increases upon inclusion of RG improvement as in the present work. The increase determined in [90] is larger than in the present parametrization and yields $\theta \simeq 3$ which is remarkably close to results from simulations based on Regge calculus [91, 92].

The present parametrization with $|\beta| \to \infty$ is also loosely related to unimodular gravity, as the conformal mode is effectively removed from the fluctuation spectrum. Still, differences to unimodular gravity remain in the gauge-fixing and ghost sector as unimodular gravity is only invariant under transversal diffeomorphisms. It is nevertheless interesting to observe that corresponding FRG calculations yield critical exponents of comparable size [93, 94].

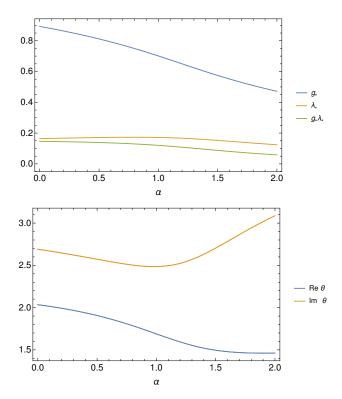


Figure 5. Linear split with field redefinition: dependence of estimates for the fixed-point values (upper panel) and the critical exponents (lower panel) on the gauge parameter α and harmonic gauge condition $\beta=1$. No qualitative and only minor quantitative differences are found for the Feynman gauge $\alpha=1$ in comparison to the Landau gauge $\alpha=0$.

In fact, the present parametrization allows for a closed form solution of the RG flow as will be presented in Sect. IV G.

E. Landau vs. Feynman gauge

Many of the pioneering computations in quantum gravity have been and still are performed within the harmonic gauge $\beta = 1$ and with $\alpha = 1$ corresponding to Feynman gauge. This is because this choice leads to a number of technical simplifications such as the direct diagonalization of the scalar modes as is visible from the off-diagonal terms in Eq. (19). Concentrating on the linear split with field redefinition, we study the α dependence for the harmonic gauge $\beta = 1$ in the vicinity of the Landau and Feynman gauges. The results for the non-Gaußian fixed point values are shown in the upper panel of Fig. 5. In essence, the fixed-point values show only a mild variation during the transition from the Landau gauge $\alpha = 0$ to the Feynman gauge $\alpha = 1$. In particular, the decrease of g_* is slightly compensated for by a mild increase of λ_* . Effectively, the observed variation is only on a level which is quantitatively similar to other parametrization dependencies, cf. Table I.

A similar conclusion holds for the more sensitive critical exponents. Real and imaginary parts of the complex pair are shown in the lower panel of Fig. 5. Starting from larger values of α , it is interesting to observe that the imaginary part Im θ decreases with decreasing α . This may be taken as an indication for a tendency towards purely real exponents; however, at about $\alpha=1$ this tendency is inverted and the exponents remain a complex pair in between Feynman gauge and Landau gauge within the present estimate.

In summary, we observe no substantial difference between the results in Feynman gauge $\alpha=1$ and those of Landau gauge $\alpha=0$ in any of the quantities of interest for the linear split and with field redefinition. Our results show an even milder dependence on the gauge parameter in comparison to the pioneering study of Ref. [72], where the regulator was chosen such as to explicitly lift the degeneracy in the sector of scalar modes in the limit $\alpha \to 0$. The present parametrization hence shows a remarkable degree of robustness against deformations away from the a-priori preferable Landau gauge. Hence, we conclude that the use of Feynman gauge is a legitimate option to reduce the complexity of computations.

F. Generalized parametrizations

Having focused so far mainly on the gauge-parameter dependencies for fixed values of the split parameter τ , we now explore the one-parameter family of parametrizations for general τ . For this, we use the Landau gauge $\alpha=0$ and take the limit $|\beta|\to\infty$, where the fixed-point estimates of all parametrizations used so far showed a large degree of stability. Figure 6 exhibits the results for the non-Gaußian fixed-point values (upper panel) and the corresponding critical exponents (lower panel).

A comparison of the results for $\tau=0$ and $\tau=1$ reveals the differences already discussed above: an increase of the fixed-point values and the occurrence of real critical exponents for the exponential split $\tau=1$. From the perspective of the principle of minimum sensitivity, it is interesting to observe that the fixed-point values develop extrema near $\tau\simeq 1.22$. The product $g_*\lambda_*$ is maximal for $\tau=1+\frac{\sqrt{3}}{24}\left(\frac{278}{\pi}\right)^{1/4}$. Also for this parametrization, the critical exponents of the fixed point are real and still close to the values for the exponential split, cf. Table I. For even larger values of τ , the critical exponents form complex pairs again.

To summarize, in the full three-parameter space defined by τ , β and $\alpha \geq 0$, we find a local extremum, i.e., a point of minimum sensitivity, at $\alpha = 0$, $\beta \to \infty$ and τ near the exponential split value $\tau = 1$. From this a-posteriori perspective, our results suggest that the exponential split (with field redefinition) in the limit where the scalar sector is represented by the σ mode may be viewed as a "best estimate" for the UV behavior of quantum Einstein gravity. Of course, due to the limitations imposed by the simplicity of our truncation, this conclu-

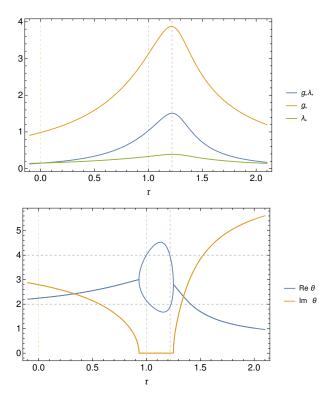


Figure 6. Parametrization dependence of fixed-point values (upper panel) and critical exponents (lower panel) as a function of the split parameter τ for the Landau gauge $\alpha=0$ and $|\beta|\to\infty$. The fixed-point values exhibit extrema near $\tau\simeq 1.22$, for the product of fixed-point values, this occurs at $\tau=1+\frac{\sqrt{3}}{24}\left(\frac{278}{\pi}\right)^{1/4}$ (red dashed vertical line). In this regime, the critical exponents are real and close to their values for the exponential split $\tau=1$ (green dashed vertical line).

sion should be taken with reservations. The resulting RG flow for $\tau=1$ is in fact remarkably simple and will be discussed next.

G. Analytical solution for the phase diagram

Let us now analyze more explicitly the results for the RG flow for the exponential split with field redefinition in the Landau gauge and in the limit $|\beta| \to \infty$. Several simplifications arise in this case. The exponential split removes any dependence of the transverse traceless and vector components of the propagator on the cosmological constant. The remaining dependence on λ in the conformal mode is finally removed by the limit $|\beta| \to \infty$. As a consequence, the cosmological constant does not couple into the flows of the Newton coupling nor into any other higher-order coupling. Still, the cosmological constant is driven by graviton fluctuations. As emphasized above, any remaining gauge dependence on the gauge parameter α drops out of the flow equations. For the RG flow of Newton coupling and cosmological constant, we find the

simple set of equations:

$$\dot{g} \equiv \beta_g = 2g - \frac{135g^2}{72\pi - 5g} \tag{34}$$

$$\dot{\lambda} \equiv \beta_{\lambda} = \left(-2 - \frac{135g}{72\pi - 5g}\right) \lambda - g\left(\frac{43}{4\pi} - \frac{810}{72\pi - 5g}\right).$$
 (35)

In addition to the Gaußian fixed point, these flow equations support a fixed point at

$$g_* = \frac{144\pi}{145}, \quad \lambda_* = \frac{48}{145}, \quad g_*\lambda_* = \frac{6912\pi}{21025},$$
 (36)

cf. Table I. Also the critical exponents θ_i being (-1) times the eigenvalues of the stability matrix $\partial \beta_{(g,\lambda)}/\partial(g,\lambda)$ can be determined analytically,

$$\theta_0 = 4, \quad \theta_1 = \frac{58}{27}.$$
 (37)

The fact that the largest critical exponent corresponds to the power-counting canonical dimension of the cosmological term is a straightforward consequence of the structure of the flow equations within this parametrization: as we have $\dot{g}=(2+\eta(g))g$ and $\dot{\lambda}=(-2+\eta(g))\lambda+\mathcal{O}(g)$, the existence of a non-Gaußian fixed point requires $\eta(g_*)=-2$. As the stability matrix is triangular, the eigenvalue associated with the cosmological term must be -4 and thus $\theta_0=4$. Rather generically, other parametrizations lead to a dependence of η also on λ and thus to a more involved stability matrix.

In the physically relevant domain of positive gravitational coupling g>0, the fixed point g_* separates a "weak" coupling phase with $g< g_*$ from a "strong" coupling phase $g>g_*$. Only the former allows for trajectories that can be interconnected with a classical regime where the dimensionless g and λ scale classically, i.e., $\dot{g}\simeq 2g$ and $\dot{\lambda}\simeq -2\lambda$ such that their dimensionful counterparts approach their observed values. Trajectories in the strong-coupling phase run to larger values of g and terminate in a singularity of β_g at $g_{\rm sing}=72\pi/5$ indicating the break-down of the truncation.

All trajectories in the weak coupling phase with $q < q_*$ run towards the Gaussian fixed point for q and thus, also the flow of λ in the infrared is dominated by the Gaussian fixed point. This implies that all trajectories emanating from the non-Gaussian fixed point with $g \leq g_*$ can be continued to arbitrarily low scales, i.e., are infrared complete. They can thus be labeled by their deep infrared value of $g\lambda$ approaching a constant, which may be identified with the product of Newton coupling and cosmological constant as observed at present. A plot of the resulting RG flow in the plane $(q, q\lambda)$ is shown in Fig. 7. It represents a global phase diagram of quantum gravity as obtained in the present truncation/parametrization. We emphasize that no singularities appear towards the IR contrary to conventional single-metric calculations based on the linear split.

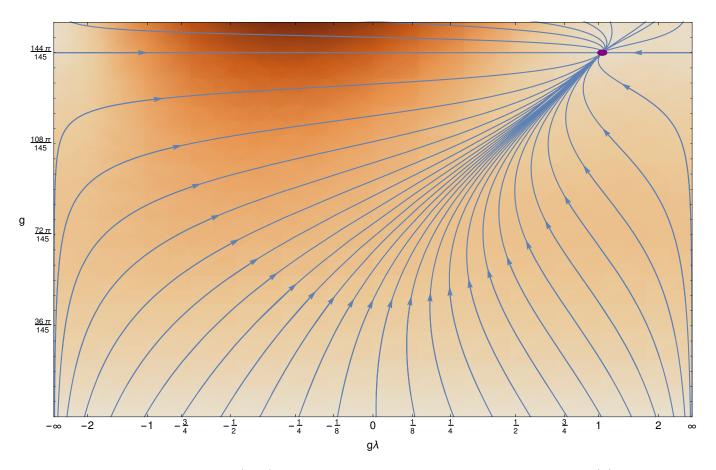


Figure 7. Global phase diagram in the $(g, g\lambda)$ plane for the exponential split with field redefinition and $|\beta| \to \infty$. Arrows point from IR to UV indicating the approach to the UV fixed point at $g_* = 144\pi/145$ and $\lambda_* = 48/145$. The color indicates a measure for the flow velocity, $(\partial_t g)^2 + (\partial_t (g\lambda/\sqrt{1+g^2\lambda^2}))^2$.

The flows (34) and (35) can be integrated analytically. Converting back to dimensionful couplings, the flow of the running Newton coupling G(k) satisfies the implicit equation,

$$G_N = \frac{G(k)}{\left(1 - \frac{145}{144\pi} k^2 G(k)\right)^{\frac{27}{29}}},\tag{38}$$

where G_N is the Newton coupling measured in the deep infrared $k \to 0$. Expanding the solution at low scales about the Newton coupling yields

$$G(k) \simeq G_N \left(1 - \frac{15}{16\pi} k^2 G_N + \mathcal{O}((k^2 G_N)^2) \right)$$
 (39)

exhibiting the anti-screening property of gravity.

The flow of the dimensionful running cosmological constant $\Lambda(k)$ can be given explicitly in terms of that of the running Newton coupling,

$$\Lambda(k) = \frac{162k^2}{25} - \frac{43G(k)k^4}{16\pi} + \ell k^2 \left(144\pi - 145G(k)k^2\right)^{\frac{25}{29}} - \frac{144\pi}{3625G(k)} \left(87 + 25\ell \left(144\pi - 145G(k)k^2\right)^{\frac{25}{29}}\right). \tag{40}$$

Here, $\ell=-\frac{29}{86400}\left(2^{-13}3^{-21}\pi^{-54}\right)^{\frac{1}{29}}\left(125\Lambda G_N+432\pi\right)$, and Λ is the value of the classical cosmological constant in the deep infrared $k\to 0$. The low-scale expansion about k=0 yields

$$\Lambda(k) \simeq \Lambda \left(1 - \frac{15}{16\pi} k^2 G_N + \mathcal{O}\left(\frac{k^4}{\Lambda^2} \Lambda G_N, (k^2 G_N)^2\right) \right)$$

Thus, $\Lambda(k)/G(k) = \Lambda/G_N + \mathcal{O}(k^4)$, implying a comparatively slow running of the ratio towards the UV. This explicit solution of the RG flow might be useful for an analysis of "RG-improved" cosmologies along the lines of [95–101].

H. Generalized ultra-local parametrizations

For the most general, ultra-local parametrization (13), it turns out that the flow equation in our truncation does only depend on the linear combinations $T_1 := \tau/4 + \tau_3$ and $T_2 := \tau_2/4 + \tau_4$, leaving only two independent split parameters. Instead of exploring the full high-dimensional parameter space, we try to identify relevant

points as inspired by our preceding results. For instance for the choice $T_1 = 1/4$ and $T_2 = -1/8$, any dependence on α drops out, indicating an enhanced insensitivity to the gauge choice. The resulting flow equations are

$$\dot{g} = 2g + \frac{135(\beta - 3)g^2}{(5\beta - 3)g - 72(\beta - 3)\pi},$$

$$\dot{\lambda} = -2\lambda + \frac{g((-669 + 215\beta)g + 36(\beta - 3)\pi(4 - 15\lambda))}{4\pi((3 - 5\beta)g + 72(\beta - 3)\pi)}.$$
(43)

In the limit $|\beta|\to\infty$, these are identical to the exponential split in the same limit. The non-Gaußian fixed point occurs at

$$g_* = \frac{144\pi(\beta - 3)}{145\beta - 411}, \quad \lambda_* = \frac{48(\beta - 3)}{145\beta - 411},$$
$$g_*\lambda_* = 6912\pi \left(\frac{\beta - 3}{145\beta - 411}\right)^2. \tag{44}$$

Apart from the pathological choice $\beta_{\text{sing}} = 3$ (incomplete gauge fixing) where this fixed point merges with the Gaussian fixed point, no further extremal point is observed except for the limit $|\beta| \to \infty$. The critical exponents are

$$\theta_0 = 4, \quad \theta_1 = \frac{58}{27} + \frac{16}{45(\beta - 3)}.$$
 (45)

Also the exponents become minimally sensitive to the choice of β for $|\beta| \to \infty$.

As an oddity, we mention the particular case $\beta = 3/5$, where the flow equations acquire a pure one-loop form. In this case, the second critical exponent is exactly 2 as it must.

More importantly, the interdependence of gauge and parametrization choices is also visible in the following fact: we observe that the choice of the gauge parameter $|\beta| \to \infty$ removes any dependence of our flow on the parameter T_2 independently of the value of α . In other words, this limit brings us back exactly to the case which we discussed above in Sect. IV F, such that the seemingly much larger class of parametrizations (13) collapses to a one-parameter family.

I. Arbitrary dimensions

Finally, we discuss the stability of the UV fixed-point scenario and its parametrization dependence in arbitrary dimensions, focusing on d>2 (for a discussion of d=2 in the present context, see [15, 17, 89]). In fact, there are some indications in the literature that the parametrization dependence is pronounced in higher dimensions. Whereas standard calculations based on the linear split generically find a UV fixed point in any dimension d>2 and gauge-fixing parameter α , see e.g. [102, 103], a recent refined choice of the parametrization to remove gauge-parameter dependence on the semi-classical level arrives

at a different result [89, 90]: the UV fixed point can be removed from the physical region if the number of physical gravity degrees of freedom becomes too large. As the latter increases with the dimensionality, there is a critical value $d_{\rm cr}$ above which asymptotically safe gravity does not exist. The resulting scenario is in line with the picture of paramagnetic dominance [104, 105], which is also at work for the QED and QCD β functions: the dominant sign of the β function coefficient arises from the paramagnetic terms in the Hessian which can be reversed if too many diamagnetically coupled degrees of freedom contribute.

Our results extend straightforwardly to arbitrary dimensions. Starting, for instance, with the most general parametrization (13) in d dimensions, the flows of g and λ depend only on the linear combinations $T_1 = \tau/d + \tau_3$ and $T_2 = \tau_2/d + \tau_4$. Comparable results as in d = 4 dimensions apply: in the limit of $|\beta| \to \infty$, also T_2 drops out such that a one-parameter family remains. In turn, a complete independence of the gauge parameter α can be realized with the parametrization specified by $T_1 = 1/d$ and $T_2 = -1/(2d)$.

We illustrate the stability properties of the asymptoticsafety scenario in arbitrary dimensions by choosing the Landau-gauge limit $\alpha \to 0$ as well as $|\beta| \to \infty$, keeping T_1 as a free parameter. Then, we know a priori that $T_1 = 1/d$ would be a preferred choice from the view point of gauge invariance; it would also correspond to the exponential parametrization $\tau = 1, \tau_3 = 0$. Fig. 8 displays the fixed-point values for $g_*\lambda_*$ as a function of T_1 for various dimensions d = 3, ..., 7. While d = 3 exhibits a rather small parametrization dependence, d=4 reproduces the earlier results of Fig. 6 (upper panel) now as a function of T_1 with an extremum not far above $T_1 = 1/4$. By contrast, $g_*\lambda_*$ develops a kink for d=5 that turns into a singularity for d=6 and larger. For increasing d, the kink approaches the preferred parametrization $T_1 = 1/d$ (vertical dashed lines in Fig. 8). The singularity in $g_*\lambda_*$ occurs for a critical dimension $d_{\rm cr} \simeq 5.731$.

This observation suggests the following interpretation: whereas we can identify a UV fixed point for any dimension as long as we choose T_1 sufficiently far away from $T_1=1/d$, we find a stable fixed-point scenario only for d=3 and d=4 integer dimensions. Already for d=5, the fixed-point product $g_*\lambda_*$ can change by two orders of magnitude by varying the parametrization, which is at least a signature for the insufficiency of the truncation. For $d \geq d_{\rm cr} \simeq 5.731$, $g_*\lambda_*$ can become unboundedly large as a function of the parametrization, signaling the instability of the fixed point.

If these features persist also beyond our truncation, they suggest that the asymptotic safety scenario may not exist far beyond the spacetime dimension d=4. Whereas this does not offer a dynamical explanation of our spacetime dimension, it may serve to rule out the mutual co-existence of extra dimensions and asymptotically safe quantum gravity.

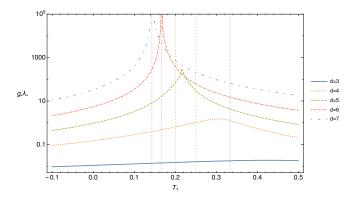


Figure 8. Parametrization dependence of fixed-point value for $g_*\lambda_*$ as a function of the split parameter T_1 in the Landau gauge $\alpha=0$ and $|\beta|\to\infty$ for different dimensions d=3,4,5,6,7 (from bottom to top). Vertical lines mark the value of the parameter $T_1=1/d$ preferred by gauge-parameter α independence. For $d\geq d_{\rm cr}\simeq 5.731$, the fixed-point product $g_*\lambda_*$ develops a singularity at $T_1=1/d$.

V. CONCLUSIONS

We have reexamined generalized parametrization dependencies of non-perturbative computations in quantum gravity based on the functional renormalization group. Whereas parametrically-ordered expansion schemes such as perturbation theory for on-shell quantities are free from such dependencies, off-shell quantities and nonperturbative expansions rather generically exhibit dependencies on, e.g., the choice of the regularization, the gauge fixing or the field parametrization. In this work, we have dealt with these dependencies in a pragmatic manner, analyzing the sensitivity and stability of the UV behavior of metric quantum gravity with respect to variations of such generalized parametrizations. We have focused on the question of the existence and the properties of a non-Gaußian UV fixed point, facilitating metric quantum gravity to be asymptotically safe. We have also concentrated on a widely studied and rather well-understood computing scheme, the Einstein-Hilbert truncation in the single-metric formulation.

Our results show a remarkable stability in a variety of qualitative aspects: for all parametrizations that exhibit rather large stationary regimes in the space of all parameters, we have found a non-Gaußian UV fixed point with two RG relevant directions, corresponding to the Newton coupling and the cosmological constant being physical parameters. For most parametrizations, the universal quantities show a remarkably mild (given the simplicity of the approximation) variation and thus a high degree of stability. Our scan of parametrization dependencies can also help identifying less robust parametrizations, and thus help judging the physical relevance of results.

Some features, however, appear to depend more strongly on the parametrization or are even visible only in specific parametrizations. Moreover, a nontrivial in-

terplay between various aspects of parametrizations, e.g., gauge choice vs. field parametrization, can arise. With hind sight, the results obtained within the exponential split with field redefinition in the limit where the graviton degrees of freedom are spanned by the transverse traceless and a scalar mode ($|\beta| \to \infty$) exhibit the highest degree of comprehensiveness: complete independence of the gauge parameter α , fully analytical and integrable global flows with a classical IR limit in the physical parameter regime, real critical exponents at the UV fixed point, and the existence of an upper critical dimension for the asymptotic safety scenario. The exploration of higher-order truncations [106] and the inclusion of matter degrees of freedom [107–109] in this parametrization appears highly worthwhile, c.f. [17, 18] for scalar matter.

In summary, our work exemplifies that a careful investigation of parametrization dependencies facilitates both a test of the robustness of nonperturbative quantum gravity computations as well as the identification of a parametrizations which may be better adapted to the physical mechanisms.

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Appendix A: Flow equations

In this section, we display the right hand side of the Wetterich equation for general Regulators $R_k[\Delta]$ and in dimension d=4. For simplicity, we introduce the anomalous dimension $\eta=(\dot{g}-2g)/g$, and refer to terms linear in η as " η -terms". Let us start with the contribution from the TT-mode:

$$S^{\text{TT}} = \frac{5}{2} Q_2 \left[\frac{\dot{R}_k - \eta R_k}{\Delta + R_k - 2\lambda (1 - \tau)} \right] - \frac{5}{12} R \left(Q_1 \left[\frac{\dot{R}_k - \eta R_k}{\Delta + R_k - 2\lambda (1 - \tau)} \right] + (4 - 3\tau) Q_2 \left[\frac{\dot{R}_k - \eta R_k}{(\Delta + R_k - 2\lambda (1 - \tau))^2} \right] \right),$$
(A1)

where the Q functionals are defined in terms of Mellin transforms [20]. For the transverse vector, and without field redefinition, let us define

$$\mathcal{G}_n^{\text{1T}} = \left[-(\dot{R}_k - \eta R_k) \left(2\lambda(1 - \tau) - \frac{1}{\alpha} (R_k + 2\Delta) \right) \right]$$

(A8)

$$-2(\dot{\lambda} + 2\lambda)(1 - \tau)R_k \bigg] \times (A2)^{-1} \left((\Delta + R_k) \left(\frac{\Delta + R_k}{\alpha} + 2\lambda(1 - \tau) \right) \right)^{-n} .$$

With that, we have

$$\begin{split} \mathcal{S}^{1\mathrm{T}} &= \frac{3}{2} Q_2 \left[\mathcal{G}_1^{1\mathrm{T}} \right] + R \left(\frac{1}{4} Q_1 \left[\mathcal{G}_1^{1\mathrm{T}} \right] \right. \\ &+ \frac{3}{2} (1 - \alpha (1 - \tau)) Q_3 \left[\mathcal{G}_2^{1\mathrm{T}} \right] - \frac{3}{4} \lambda (1 - \tau) Q_2 \left[\mathcal{G}_2^{1\mathrm{T}} \right] \right) \,. \end{split}$$

On the other hand, the contribution with field redefinition reads.

$$S_{\text{fr}}^{1\text{T}} = \frac{3}{2} Q_2 \left[\frac{\dot{R}_k - \eta R_k}{\Delta + R_k - 2\alpha\lambda(1 - \tau)} \right]$$

$$+ R \left(\frac{1}{8} Q_1 \left[\frac{\dot{R}_k - \eta R_k}{\Delta + R_k - 2\alpha\lambda(1 - \tau)} \right] \right)$$

$$+ \frac{3}{8} (1 - 2\alpha(1 - \tau)) Q_2 \left[\frac{\dot{R}_k - \eta R_k}{(\Delta + R_k - 2\alpha\lambda(1 - \tau))^2} \right] \right) .$$
(A4)

For the scalar contribution, we first define

$$\pi^{\sigma} = -\frac{3}{4\alpha} \left(4\alpha\lambda (1-\tau)(\Delta + R_k)^2 + (\alpha - 3)(\Delta + R_k)^3 \right)$$
(A5)

$$\pi^{\rm h} = -\frac{1}{16\alpha} \left(-4\alpha\lambda(1+\tau) + (3\alpha - \beta^2)(\Delta + R_k) \right)$$
 (A6)

$$\pi^{\mathbf{x}} = -\frac{3}{8\alpha}(\alpha - \beta)(\Delta + R_k)^2 \tag{A7}$$

$$\rho^{\sigma} = -\frac{3}{4\alpha} \left(4\alpha (\dot{\lambda} + (2 - \eta)\lambda)(1 - \tau)(2\Delta + R_k)R_k + 8\alpha\lambda(1 - \tau)(\Delta + R_k)\dot{R}_k + 3(\alpha - 3)(\Delta + R_k)^2\dot{R}_k - (\alpha - 3)(3\Delta^2 + 3\Delta R_k + R_k^2)\eta R_k \right)$$
(A8)

$$\rho^{h} = -\frac{1}{16\alpha} (3\alpha - \beta^{2})(\dot{R}_{k} - \eta R_{k}) \tag{A9}$$

$$\rho^{\mathbf{x}} = -\frac{3}{8\alpha}(\alpha - \beta)(2\dot{R}_k(\Delta + R_k) - \eta R_k(2\Delta + R_k)). \tag{A10}$$

The contribution is

$$S^{\sigma h} = \frac{1}{2} Q_{2} \left[\frac{\pi^{\sigma} \rho^{h} + \pi^{h} \rho^{\sigma} - 2\pi^{x} \rho^{x}}{\pi^{\sigma} \pi^{h} - (\pi^{x})^{2}} \right] + R \left\{ \frac{1}{12} Q_{1} \left[\frac{\pi^{\sigma} \rho^{h} + \pi^{h} \rho^{\sigma} - 2\pi^{x} \rho^{x}}{\pi^{\sigma} \pi^{h} - (\pi^{x})^{2}} \right] - \frac{3}{4\alpha} (6 - \alpha(4 - 3\tau)) Q_{4} \left[\frac{\rho^{h}}{\pi^{\sigma} \pi^{h} - (\pi^{x})^{2}} \right] + \lambda(1 - \tau) Q_{3} \left[\frac{\rho^{h}}{\pi^{\sigma} \pi^{h} - (\pi^{x})^{2}} \right] - \frac{\tau}{32} Q_{2} \left[\frac{\rho^{\sigma}}{\pi^{\sigma} \pi^{h} - (\pi^{x})^{2}} \right] - \frac{\alpha - \beta}{4\alpha} Q_{3} \left[\frac{\rho^{x}}{\pi^{\sigma} \pi^{h} - (\pi^{x})^{2}} \right] + \frac{3}{4\alpha} (6 - \alpha(4 - 3\tau)) Q_{4} \left[\frac{\pi^{h} \left(\pi^{\sigma} \rho^{h} + \pi^{h} \rho^{\sigma} - 2\pi^{x} \rho^{x} \right)}{\left(\pi^{\sigma} \pi^{h} - (\pi^{x})^{2} \right)^{2}} \right] + \lambda(1 - \tau) Q_{3} \left[\frac{\pi^{h} \left(\pi^{\sigma} \rho^{h} + \pi^{h} \rho^{\sigma} - 2\pi^{x} \rho^{x} \right)}{\left(\pi^{\sigma} \pi^{h} - (\pi^{x})^{2} \right)^{2}} \right] + \frac{\tau}{32} Q_{2} \left[\frac{\pi^{\sigma} \left(\pi^{\sigma} \rho^{h} + \pi^{h} \rho^{\sigma} - 2\pi^{x} \rho^{x} \right)}{\left(\pi^{\sigma} \pi^{h} - (\pi^{x})^{2} \right)^{2}} \right] + \frac{\alpha - \beta}{16\alpha} Q_{2} \left[\frac{\pi^{x} \left(\pi^{\sigma} \rho^{h} + \pi^{h} \rho^{\sigma} - 2\pi^{x} \rho^{x} \right)}{\left(\pi^{\sigma} \pi^{h} - (\pi^{x})^{2} \right)^{2}} \right] \right\}.$$
(A11)

With field redefinition, define

$$\pi_{\rm fr}^{\sigma} = -\frac{3}{4\alpha} \left(4\alpha\lambda(1-\tau) + (\alpha-3)(\Delta + R_k) \right) \tag{A12}$$

$$\pi_{\rm fr}^{\rm h} = -\frac{1}{16\alpha} \left(-4\alpha\lambda(1+\tau) + (3\alpha - \beta^2)(\Delta + R_k) \right) \tag{A13}$$

$$\pi_{\rm fr}^{\rm x} = -\frac{3}{8\alpha}(\alpha - \beta)(\Delta + R_k) \tag{A14}$$

$$\rho_{\rm fr}^{\sigma} = \frac{3}{4\alpha} (3 - \alpha)(\dot{R}_k - \eta R_k) \tag{A15}$$

$$\rho_{\rm fr}^{\rm h} = -\frac{1}{16\alpha} (3\alpha - \beta^2)(\dot{R}_k - \eta R_k) \tag{A16}$$

$$\rho_{\rm fr}^{\rm x} = -\frac{3}{8\alpha}(\alpha - \beta)(\dot{R}_k - \eta R_k). \tag{A17}$$

Then, the scalar contribution is

$$S_{\rm fr}^{\sigma \rm h} = \frac{1}{2} Q_2 \left[\frac{\pi_{\rm fr}^{\sigma} \rho_{\rm fr}^{\rm h} + \pi_{\rm fr}^{\rm h} \rho_{\rm fr}^{\sigma} - 2\pi_{\rm fr}^{\rm x} \rho_{\rm fr}^{\rm x}}{\pi_{\rm fr}^{\sigma} \pi_{\rm fr}^{\rm h} - (\pi_{\rm fr}^{\rm x})^2} \right] + R \left\{ \frac{1}{12} Q_1 \left[\frac{\pi_{\rm fr}^{\sigma} \rho_{\rm fr}^{\rm h} + \pi_{\rm fr}^{\rm h} \rho_{\rm fr}^{\sigma} - 2\pi_{\rm fr}^{\rm x} \rho_{\rm fr}^{\rm x}}{\pi_{\rm fr}^{\sigma} \pi_{\rm fr}^{\rm h} - (\pi_{\rm fr}^{\rm x})^2} \right] - \frac{3}{8\alpha} (1 - \alpha (1 - \tau)) Q_2 \left[\frac{\rho_{\rm fr}^{\rm h}}{\pi_{\rm fr}^{\sigma} \pi_{\rm fr}^{\rm h} - (\pi_{\rm fr}^{\rm x})^2} \right] \right\}$$

$$-\frac{\tau}{32}Q_{2}\left[\frac{\rho_{\mathrm{fr}}^{\sigma}}{\pi_{\mathrm{fr}}^{\sigma}\pi_{\mathrm{fr}}^{h} - (\pi_{\mathrm{fr}}^{\mathrm{x}})^{2}}\right] - \frac{\alpha - \beta}{16\alpha}Q_{2}\left[\frac{\rho_{\mathrm{fr}}^{\mathrm{x}}}{\pi_{\mathrm{fr}}^{\sigma}\pi_{\mathrm{fr}}^{h} - (\pi_{\mathrm{fr}}^{\mathrm{x}})^{2}}\right] + \frac{3}{8\alpha}(1 - \alpha(1 - \tau))Q_{2}\left[\frac{\pi_{\mathrm{fr}}^{h}\left(\pi_{\mathrm{fr}}^{\sigma}\rho_{\mathrm{fr}}^{h} + \pi_{\mathrm{fr}}^{h}\rho_{\mathrm{fr}}^{\sigma} - 2\pi_{\mathrm{fr}}^{\mathrm{x}}\rho_{\mathrm{fr}}^{\mathrm{x}}\right)}{\left(\pi_{\mathrm{fr}}^{\sigma}\pi_{\mathrm{fr}}^{h} - (\pi_{\mathrm{fr}}^{\mathrm{x}})^{2}\right)^{2}}\right] + \frac{1}{8\alpha}(1 - \alpha(1 - \tau))Q_{2}\left[\frac{\pi_{\mathrm{fr}}^{h}\left(\pi_{\mathrm{fr}}^{\sigma}\rho_{\mathrm{fr}}^{h} + \pi_{\mathrm{fr}}^{h}\rho_{\mathrm{fr}}^{\sigma} - 2\pi_{\mathrm{fr}}^{\mathrm{x}}\rho_{\mathrm{fr}}^{\mathrm{x}}\right)}{\left(\pi_{\mathrm{fr}}^{\sigma}\pi_{\mathrm{fr}}^{h} - (\pi_{\mathrm{fr}}^{\mathrm{x}})^{2}\right)^{2}}\right] + \frac{1}{8\alpha}(1 - \alpha(1 - \tau))Q_{2}\left[\frac{\pi_{\mathrm{fr}}^{h}\left(\pi_{\mathrm{fr}}^{\sigma}\rho_{\mathrm{fr}}^{h} + \pi_{\mathrm{fr}}^{h}\rho_{\mathrm{fr}}^{\sigma} - 2\pi_{\mathrm{fr}}^{\mathrm{x}}\rho_{\mathrm{fr}}^{\mathrm{x}}\right)}{\left(\pi_{\mathrm{fr}}^{\sigma}\pi_{\mathrm{fr}}^{h} - (\pi_{\mathrm{fr}}^{\mathrm{x}})^{2}\right)^{2}}\right] + \frac{1}{8\alpha}(1 - \alpha(1 - \tau))Q_{2}\left[\frac{\pi_{\mathrm{fr}}^{h}\left(\pi_{\mathrm{fr}}^{\sigma}\rho_{\mathrm{fr}}^{h} + \pi_{\mathrm{fr}}^{h}\rho_{\mathrm{fr}}^{\sigma} - 2\pi_{\mathrm{fr}}^{\mathrm{x}}\rho_{\mathrm{fr}}^{\mathrm{x}}\right)}{\left(\pi_{\mathrm{fr}}^{\sigma}\pi_{\mathrm{fr}}^{h} - (\pi_{\mathrm{fr}}^{\mathrm{x}})^{2}\right)^{2}}\right] + \frac{1}{8\alpha}(1 - \alpha(1 - \tau))Q_{2}\left[\frac{\pi_{\mathrm{fr}}^{h}\left(\pi_{\mathrm{fr}}^{\sigma}\rho_{\mathrm{fr}}^{h} + \pi_{\mathrm{fr}}^{h}\rho_{\mathrm{fr}}^{\sigma} - 2\pi_{\mathrm{fr}}^{\mathrm{x}}\rho_{\mathrm{fr}}^{\mathrm{x}}\right)}{\left(\pi_{\mathrm{fr}}^{\sigma}\pi_{\mathrm{fr}}^{h} - (\pi_{\mathrm{fr}}^{\mathrm{x}})^{2}\right)^{2}}\right] + \frac{3}{8\alpha}(1 - \alpha(1 - \tau))Q_{2}\left[\frac{\pi_{\mathrm{fr}}^{h}\left(\pi_{\mathrm{fr}}^{\sigma}\rho_{\mathrm{fr}}^{h} + \pi_{\mathrm{fr}}^{h}\rho_{\mathrm{fr}}^{\sigma} - 2\pi_{\mathrm{fr}}^{\mathrm{x}}\rho_{\mathrm{fr}}^{\mathrm{x}}\right)}{\left(\pi_{\mathrm{fr}}^{\sigma}\pi_{\mathrm{fr}}^{h} - (\pi_{\mathrm{fr}}^{\mathrm{x}})^{2}\right)^{2}}\right]$$

Further, the ghost contribution reads without field redefinition,

$$S^{\text{gh}} = -5Q_2 \left[\frac{\dot{R}_k}{\Delta + R_k} \right] - R \left(\frac{7}{12} Q_1 \left[\frac{\dot{R}_k}{\Delta + R_k} \right] \right)$$

$$+ \frac{3}{4} Q_2 \left[\frac{\dot{R}_k}{(\Delta + R_k)^2} \right] + \frac{4}{3 - \beta} Q_3 \left[\frac{\dot{R}_k}{(\Delta + R_k)^3} \right] .$$
(A19)

With field redefinition, it is

$$S_{\rm fr}^{\rm gh} = -4Q_2 \left[\frac{\dot{R}_k}{\Delta + R_k} \right] - R \left(\frac{5}{12} Q_1 \left[\frac{\dot{R}_k}{\Delta + R_k} \right] \right)$$

 $+ \left(\frac{3}{4} + \frac{1}{3-\beta}\right) Q_2 \left[\frac{\dot{R}_k}{(\Delta + R_k)^2}\right] \right) . \quad (A20)$

Finally, the contribution of the Jacobian for the case without field redefinition is

$$S^{\text{Jac}} = \frac{1}{2} S^{\text{gh}} \Big|_{\beta=0} + Q_2 \left[\frac{\dot{R}_k}{\Delta + R_k} \right] + \frac{1}{6} R Q_1 \left[\frac{\dot{R}_k}{(\Delta + R_k)^2} \right]. \tag{A21}$$

With field redefinition, all Jacobians cancel, at least on maximally symmetric backgrounds, which is sufficient for the truncation considered here [72].

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